Radial Symmetry of Solutions of System of Nonlinear Elliptic Boundary Value Problems

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Abstract
Radial symmetry of positive solutions of system of nonlinear elliptic bound- ary value problems in $\mathbb{R}^n$ is studied. We apply the moving plane method based on maximum principle to obtain our result of symmetry of solutions on unbounded domain $\mathbb{R}^n$.

Keywords: Maximum principles, Moving plane method, Radial symmetry; System of nonlinear elliptic boundary value problems.

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1. Introduction:

$$\Delta u + f(|x|; u) = 0 \text{ in } \mathbb{R}^n$$
and
$$u(x) \to 0 \text{ as } |x| \to \infty.$$ 


$$\Delta u = uv^2; \Delta v = vu^2; u \geq 0; v \geq 0 \text{ in } \mathbb{R}^n.$$  


In this paper we study the radial symmetry of positive solutions for system of nonlinear elliptic boundary value problem in $\mathbb{R}^n$. We consider the nonlinear elliptic boundary value problem of the form

$$\Delta u + f(|x|; u; v) = 0 \quad \text{in } \mathbb{R}^n$$

$$\Delta v + g(|x|; u; v) = 0 \quad \text{and}$$

$$u(x) \to 0, \ v(x) \to 0 \text{ as } |x| \to \infty.$$  

These type of systems occur in many models of physics, where study of symmetry property is important. Our approach is based on the maximum principle in unbounded domains together with the moving plane method. This approach helps us to prove our results. We organise the paper as follows: In section 2, the preliminary results and some useful lemmas are proved. The symmetric result is proved and some illustrative examples are given in the last section.

2. Preliminaries

In this section, first we state some basic lemmas and boundary maximum principle which are useful to prove our main result.

Lemma 2.1 [12] Hopf boundary lemma: Suppose that $\Omega$ satisfies the interior sphere condition at $x_0 \in \partial \Omega$. Let $L$ be uniformly elliptic with $c(x) \leq 0$ where
Lemma 2.2 [17] Let $\Omega$ be an unbounded domain in $\mathbb{R}^n$, and let $L$ denote the uniformly elliptic differential operator of the form

$$L(u) = a^{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + b^i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

where $a^{ij}, b^i, c \in L^\infty(\Omega)$. Suppose that $u \neq 0$ satisfies $L(u) \leq 0$ in $\Omega$ and $u \geq 0$ on $\partial \Omega$.

Furthermore, suppose that there exist a function $w$ such that $w > 0$ on $\Omega \cup \partial \Omega$ and $L(w) \leq 0$ in $\Omega$.

If $\frac{u(x)}{w(x)} \to 0$ as $|x| \to \infty$, $x \in \Omega$ then $u > 0$ in $\Omega$.

Theorem 2.1 [19] Let $u(x)$ satisfies differential inequality

$$L(u) \geq 0$$

in a domain $\Omega$ where $L$ is uniformly elliptic. If there exist a function $w(x)$ such that, $w(x) > 0$ on $\Omega \cup \partial \Omega$ and satisfies the differential inequality

$$L(w) \leq 0 \text{ in } \Omega$$

then $\frac{u(x)}{w(x)}$ can not attain a non negative maximum at a point $p$ on $\partial \Omega$, which lies on the boundary of a ball in $\Omega$ and if $\frac{u(x)}{w(x)}$ is not constant then, $\frac{\partial}{\partial \nu} \left( \frac{u}{w} \right) > 0$

$$\sum_i \frac{\partial}{\partial \nu} \left( \frac{u}{w} \right) > 0 \text{ at } P.$$

where $\frac{\partial}{\partial \nu}$ is any outward directional derivative.
3. Main Results

In this section, we prove our main result. We define the plane \( T_\lambda \), for a real number \( \lambda \) as follows \( T_\lambda = \{ x : x = (x_1; x_2; x_3; \ldots ; x_n); x_1 = \lambda \} \), which is perpendicular to \( X_1 \)-axis. We will move this plane continuously normal to itself to new position till it begins to intersect \( \Omega \). After that point the plane advances in \( \Omega \) along \( X_1 \)-axis and cut of cap \( \sum_x \); which is the portion of \( \Omega \) and lies in the same side of the plane \( T_\lambda \) as the original plane \( T \). Let \( \sum_x = \{ x : x_1 < \lambda ; x \in \Omega \} \). Let \( x^1 = (2\lambda - x_1; x_2; x_3; \ldots ; x_n) \) be the reflection of the point \( x = (x_1; x_2; x_3; \ldots ; x_n) \), about the plane \( T_\lambda \). Define \( w_1; \lambda(x) = u(x) - u(x_i) \), and \( w_2; \lambda(x) = v(x) - v(x_i) \). We have \( |x^1| \geq |x| \) and \( u(x_i) < u(2\lambda - x_1; x_2; x_3; \ldots ; x_n) \). Also define set \( \Lambda = \{ \lambda \in (0; \infty) : w_1; \lambda(x) > 0; w_2; \lambda(x) > 0 \} \) for \( x \in \sum_x \).

Now, we prove our main result.

**Theorem 3.1** Suppose that

(i) \((u; v) \in C^2\) is a positive solution of the system of nonlinear elliptic boundary value problem \((1.1)-(1.2)\),

(ii) functions \( f \) and \( g \) are continuous in all its variables and \( C1 \) in \( u \geq 0, v \geq 0 \),

(iii) \( f(|x|, u(x); v(x_1; x_2; x_3; \ldots ; x_1-1; 2\lambda - x_i; x_i+1; \ldots; x_n)) = f(|x|, u(x); v(x_1; x_2; x_3; \ldots ; x_i-1; x_i; x_i+1; \ldots; x_n) \) , for all \( 1 \leq i \leq n \),

(iv) \( g(|x| ; u(x_1; x_2; x_3; \ldots ; x_i-1; 2\lambda - x_i; x_i+1; \ldots; x_n); v(x)) = g(|x| ; u(x_1; x_2; x_3; \ldots ; x_i-1; x_i; x_i+1; \ldots; x_n); v(x)) \) for all \( 1 \leq i \leq n \),

(v) functions \( f \) and \( g \) are nonincreasing in \( |x| \), for each fixed \( u \geq 0, v \geq 0 \).

Further we define \( U \), \( V \) and \( \Phi \) such as

\[
U(r) = \sup \{ u(x) : |x| \geq r \} \tag{3.1}
\]

\[
V (r) = \sup \{ v(x) : |x| \geq r \} \tag{3.2}
\]

\[
\Phi (|x|) = \sup \{ fu(|x|; u; v); gv(|x| ; u; v) : 0 \leq u(x) \leq U(r); 0 \leq v(x) \leq V (r) \} \tag{3.3}
\]

Furthermore assume that there exist positive functions \( z_1; z_2 \) on \( |x| \geq R_0 \); for some positive constant \( R_0 \) satisfying differential inequalities

\[
\Delta z_1 + \Phi(|x|)z_1 \leq 0 \quad \text{In} \quad |x| > R_0 \tag{3.4}
\]

and

\[
\Delta z_2 + \Phi(|x|)z_2 \leq 0
\]
\[
\lim_{|x| \to \infty} \frac{U(x)}{z_1(x)} = 0 
\]
(3.5)

\[
\lim_{|x| \to \infty} \frac{V(x)}{z_2(x)} = 0 
\]
(3.6)

Then \((u; v)\) is radially symmetric about some \(x_0\) in \(\mathbb{R}^n\) and \(ur < 0\), \(ur < 0\) for \(r = |x - x_0| > 0\).

To prove our result following lemmas are useful. First we prove them.

Lemma 3.1 Let \(\lambda \geq 0\), then \(w_1; \lambda(x)\) and \(w_2; \lambda(x)\) satisfies differential inequalities

\[
\Delta w_{1,\lambda}(x) + C_{1,\lambda}(x) w_{1,\lambda}(x) \leq 0 \quad \text{in} \quad \sum \iota 
\]

\[
\Delta w_{2,\lambda}(x) + C_{2,\lambda}(x) w_{2,\lambda}(x) \leq 0 \quad \text{in} \quad \sum \iota 
\]

Where

\[
C_{1,\lambda}(x) = \int_0^1 f(|x|, u(x^\lambda) + t[u(x) - u(x^\lambda)]; v(x)) dt 
\]

\[
C_{2,\lambda}(x) = \int_0^1 g(|x|, u(x^\lambda), v(x^\lambda) + t[v(x) - v(x^\lambda)]) dt 
\]

Proof: Since \(f(|x|; u; v)\) and \(g(|x|; u; v)\) are nonincreasing in \(|x|\) and \(|x^\lambda| > |x|\) for \(x \in \sum \iota\) hold. We observe that \(u(x^\lambda)\) and \(v(x^\lambda)\) satisfy the equations

\[
\Delta u(x^\lambda) + f(|x^\lambda|, u(x^\lambda), v(x^\lambda)) = 0 \quad \text{in} \quad \sum \iota 
\]
(3.7)

\[
\Delta v(x^\lambda) + g(|x^\lambda|, u(x^\lambda), v(x^\lambda)) = 0 \quad \text{in} \quad \sum \iota 
\]
(3.8)

Subtracting (3.7) from first equation of (1.1) we get

\[
\Delta w_{1,\lambda}(x) + C_{1,\lambda}(x) w_{1,\lambda}(x) \leq 0 \quad \text{in} \quad \sum \iota 
\]
where

\[
C_{1,\lambda}(x) = \int_0^1 f(|x|, u(x^\lambda) + t[u(x) - u(x^\lambda)]; v(x)) dt 
\]

Similarly subtracting (3.8) from second equation of (1.1) we can prove that,

\[
\Delta w_{2,\lambda}(x) + C_{2,\lambda}(x) w_{2,\lambda}(x) \leq 0 \quad \text{in} \quad \sum \iota 
\]
where
\[ C_{1,\lambda}(x) = \int_{0}^{1} h(|x|, u(x), v(x^+), t[v(x) - v(x^+)]) dt \]

The proof of the lemma is completed.

**Lemma 3.2** Let \( \lambda > 0 \), If \( w_{1,\lambda}(x) > 0 \), and \( w_{2,\lambda}(x) > 0 \) in \( \sum_{\lambda} \setminus \overline{B}_0 \), Then \( \lambda \in \Lambda \).

**Proof:** By using assumptions and lemma 3.1, we have

\[ \Delta w_{1,\lambda}(x) + C_{1,\lambda}(x)w_{1,\lambda}(x) \leq 0 \]

in \( \sum_{\lambda} \setminus \overline{B}_0 \)

and

\[ \Delta w_{2,\lambda}(x) + C_{2,\lambda}(x)w_{2,\lambda}(x) \leq 0 \]

Since \( U(r) \) and \( V(r) \) are nonincreasing, we have

\[ 0 \leq u(x^+ + t(u(x) - u(x^+))) \leq U(|x|) \]

\[ 0 \leq v(x^+ + t(v(x) - v(x^+))) \leq V(|x|) \]

Then we observe that

\[ C_{1,\lambda}(x) \leq \int_{0}^{1} \Phi(|x|) dt \leq \Phi(|x|) \text{ in } \sum_{\lambda} \]

Similarly we can show that,

\[ C_{2,\lambda}(x) \leq \int_{0}^{1} \Phi(|x|) dt \leq \Phi(|x|) \text{ in } \sum_{\lambda} \]

**From (3.4) we have**

\[ \Delta z_{1} + C_{1,\lambda}(x) \leq 0 \text{ in } \sum_{\lambda} \setminus \overline{B}_0 \]

\[ \Delta z_{2} + C_{2,\lambda}(x) \leq 0 \text{ in } \sum_{\lambda} \setminus \overline{B}_0 \]

and

\[ \frac{w_{1,\lambda}(x)}{z_{1}(x)} \rightarrow 0 \text{, } \frac{w_{2,\lambda}(x)}{z_{2}(x)} \rightarrow 0 \text{ as } |x| \rightarrow \infty \text{, for } x \in \sum_{\lambda} \setminus \overline{B}_0 \]

**Hence by Lemma (2.2)**

\( w_{1,\lambda}(x) > 0 \), and \( w_{2,\lambda}(x) > 0 \) in \( \sum_{\lambda} \setminus \overline{B}_0 \).

From this it follows that,

\( w_{1,\lambda}(x) > 0 \), and \( w_{2,\lambda}(x) > 0 \) in \( \sum_{\lambda} \)

Therefore \( \lambda \in \Lambda \). This completes the proof.
Lemma 3.3 Let $\lambda \in A$, then $\frac{\partial u}{\partial x_1} < 0$ and $\frac{\partial v}{\partial x_1} < 0$ on $T_\lambda$.

Proof: By Lemma 3.1 we have
\[
\Delta w_{1,\lambda}(x) + C_{1,\lambda}(x)w_{1,\lambda}(x) \leq 0 \quad \text{in} \quad \sum_{\lambda}
\]
\[
\Delta w_{2,\lambda}(x) + C_{2,\lambda}(x)w_{2,\lambda}(x) \leq 0 \quad \text{in} \quad \sum_{\lambda}
\]
Since $w_{1,\lambda}(x) = 0$ on $T_\lambda$ by Hopf’s boundary lemma we have $\frac{\partial w_{1,\lambda}(x)}{\partial x_1} < 0$ and $\frac{\partial w_{2,\lambda}(x)}{\partial x_1} < 0$ on $T_\lambda$. We have $\frac{\partial w_{2,\lambda}(x)}{\partial x_1} = 2\frac{\partial v}{\partial x_1}$ and $\frac{\partial w_{1,\lambda}(x)}{\partial x_1} = 2\frac{\partial v}{\partial x_1}$.

Therefore $\frac{\partial u}{\partial x_1} < 0 < \frac{\partial u}{\partial x_1} < 0$.

Now we prove theorem 3.1.

Proof of the theorem:

Since $u(x); v(x)$ are positive with
\[
\lim_{|x| \to 0} u(x) = 0
\]
and
\[
\lim_{|x| \to 0} v(x) = 0
\]
then there exist $R1, R2 > R0$ such that
\[
\max\{u(x) : |x| \geq R_1\} < \min\{u(x) : |x| \leq R_0\} \quad (3.9)
\]
\[
\max\{v(x) : |x| \geq R_2\} < \min\{v(x) : |x| \leq R_0\} \quad (3.10)
\]
\[
w_{1,\lambda}(x) > 0, \quad \text{in} \quad B_0
\]

Also from equation (3.10) we get,
\[
w_{2,\lambda}(x) > 0, \quad \text{in} \quad B_0
\]

Then by Lemma 3.2 we have $\lambda \in A$. Therefore $[Rm; x] \subset A$. Hence step-I is completed.

Step-II: In this step we prove that, if $\lambda_0 \in A$ then there exist $\varepsilon > 0$ such that

$(\lambda_0 - \varepsilon; \lambda_0) \subset A$. Assume to the contrary that there exist an increasing sequence $\{\lambda_i\}, i = 1; 2; 3; \ldots$ such that $\lambda_i \notin A$ and $\lambda_i \to \lambda_0$ as $i \to \infty$, which contradicts to Lemma 3.2.
Therefore we have a sequence \( \{x_i\} \) such that \( x_i \in \sum_{i \geq 0} B_i \) and \( w_{1,2i}(x_i) \leq 0 \) or \( w_{2,2i}(x_i) \leq 0 \). A subsequence \( \{x'_i\} \), converges to some point \( x_0 \in \sum_{i \geq 0} B_i \). Then \( w_{1,2i}(x_i) \leq 0 \) or \( w_{2,2i}(x_i) \leq 0 \). But in \( \sum_{i \geq 0} \), we must have \( x_i \rightarrow x_i^{2i} \) \( w_{1,2i}(x_0) > 0 \) and \( y_i \rightarrow x_0 \). Therefore we conclude that \( x_0 \in T_{x_i} \). By mean value theorem, there exist a point \( y_i \) such that

\[
\frac{\partial u}{\partial x_i}(y_i) \geq 0
\]

and

\[
\frac{\partial v}{\partial x_i}(y_i) \geq 0
\]

on the line segment joining \( x_i \rightarrow x_i^{2i} \) for each \( i = 1; 2; 3; \ldots \). Also \( y_i \rightarrow x_0 \) as \( i \rightarrow \infty \).

So \( \frac{\partial u}{\partial x_i}(x_0) \geq 0 \) and \( \frac{\partial v}{\partial x_i}(x_0) \geq 0 \).

But by Lemma 3.1 we have

\[
\frac{\partial u}{\partial x_i}(x_0) \leq 0
\]

and

\[
\frac{\partial v}{\partial x_i}(x_0) \leq 0
\]

which is a contradiction and step II is completed.

Step-III: Consider the following two statements (A) and (B),

(A) \( u(x) \equiv u(x^{2i}) \) and \( v(x) \equiv v(x^{2i}) \) for some \( \lambda_1 > 0 \) and \( \frac{\partial u}{\partial x_i} \leq 0, \frac{\partial v}{\partial x_i} \leq 0 \) on \( T_\lambda \) for \( \lambda > \lambda_i \).

Or

(B) \( u(x) \geq u(x^0) \) and \( v(x) \geq v(x^0) \) in \( \sum_{i \geq 0} \) and \( \frac{\partial u}{\partial x_i} \leq 0, \frac{\partial v}{\partial x_i} \leq 0 \) on \( T_\lambda \) for \( \lambda > 0 \).

We consider two cases (i)If \( \lambda_1 > 0 \), then we prove that statement (A) holds. (ii) If \( \lambda_1 = 0 \), then we prove that statement (B) holds.

Define \( \lambda_i = \min\{\lambda > 0 : [\lambda, \infty) \subset \Lambda\} \).
Case (i) where $\lambda_1 > 0$.
We have

\[ w_{1,\lambda_1}(x) = u(x) - u(x^1) \]
\[ w_{2,\lambda_1}(x) = v(x) - v(x^1) \]

From the continuity of $u$ and $v$, we have $w_{1,\lambda_1}(x) \geq 0, w_{2,\lambda_1}(x) \geq 0$ in $\sum_{\lambda_1}$. From lemma 3.1 it follows that

\[ \Delta w_{1,\lambda_1}(x) + c_{1,\lambda_1}(x)w_{1,\lambda_2}(x) \leq 0 \quad \text{in } \sum_{\lambda_1} \]
\[ \Delta w_{2,\lambda_1}(x) + c_{2,\lambda_1}(x)w_{2,\lambda_2}(x) \leq 0 \]

Hence by strong maximum principle we have either $w_{1,\lambda_1}(x) \geq 0, w_{2,\lambda_1}(x) \geq 0$ or $w_{1,\lambda_1}(x) = 0, w_{2,\lambda_1}(x) = 0$ in $\sum_{\lambda_1}$

Assume that $w_{1,\lambda_1}(x) > 0, w_{2,\lambda_1}(x) > 0$ in $\sum_{\lambda_1}$, then by lemma 3.2, $\lambda \in \Lambda$. From step-II there exists $\varepsilon > 0$ such that $(\lambda_1 - \varepsilon, \lambda_1] \subset \Lambda$.

This is contradiction to the definition of $\lambda_1$. Therefore $w_{1,\lambda_1}(x) = 0, w_{2,\lambda_1}(x) = 0$ in $\sum_{\lambda_1}$.

Since $(\lambda_1, \infty) \subset \Lambda$, we have $\frac{\partial u}{\partial x_1} \leq 0, \frac{\partial v}{\partial x_1} \leq 0$ on $T_\lambda$ for $\lambda > \lambda_1$, by Lemma 3.3, the statement (A) is follows for case (i).

**Case (ii)** Consider the case where $\lambda_1 = 0$. From the continuity of $u$ and $v$ we have $u(x) \geq u(x^0), v(x) \geq v(x^0)$ in $\sum_0$. By lemma 3.3 we get $\frac{\partial u}{\partial x_1} \leq 0, \frac{\partial v}{\partial x_1} \leq 0$ on $T_\lambda$ for $\lambda > 0$.

Thus statement (B) holds.

If statement (B) occurs in step (III), we can repeat the previous steps I – III for negative $X_1$-direction to conclude that either $(u, v)$ is symmetric in $X_1$-direction about some plane $x_1 = \lambda_1 < 0$. Therefore $(u, v)$ must be symmetric in $X_1$-direction about some plane and strictly decreasing away from the plane. As we may take any direction as the $X_1$-direction, we conclude that $(u, v)$ is symmetric in every direction about some plane. Therefore $(u, v)$ is radially symmetric about some point $x_0 \in R^1, \frac{\partial u}{\partial r} < 0, \frac{\partial v}{\partial r} < 0$, for $r > 0$.

**Example**
Now, we discuss an example to illustrate theorem 3.1.
Example 3.1 Consider the elliptic system
\begin{align}
\Delta u + (n-3)v &= 0 \tag{3.11} \\
\Delta v + 3(n-5)u^2 v &= 0 \tag{3.12}
\end{align}

Here \( f(|x|, u, v) = (n-3)v \) is linear and \( f(|x|, u, v) = 3(n-5)u^2 v \) is nonlinear function.

Clearly \( \Phi(|x|) = 3(n-5)u^2 = \frac{3(n-5)}{|x|} \) where \( u = \frac{1}{|x|} \). Suppose \( z_1(x) = \frac{1}{|x|^{\frac{1}{2}}} \), and \( z_2(x) = \frac{1}{|x|^{\frac{1}{2}}} \). Here \( z1(x) \) and \( z2(x) \) satisfies the inequalities (3.4).

We also have, \( \lim_{|x| \to \infty} \frac{u(x)}{z_1(x)} = 0 \)

Thus condition (3.5) is satisfied. similarly we can show that condition (3.6) holds for the function \( z_2(x) = \frac{1}{|x|^{\frac{1}{2}}} \). Thus all the conditions of theorem 3.1 are satisfied.

Therefore solutions must be radially symmetric. Clearly this system of equations have radially symmetric solutions about the origin.

4. References


